DETERMINING EQUATIONS AND SOME PROBLEMS OF THE VARIABLE-MODULUS THEORY OF ELASTICITY OF ANISOTROPIC MATERIALS

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The mechanical characteristics of different alloys, polymers, and composites differ as a result of anisotropy and their dependence of the type of stress state. These features are manifest both during instantaneous loading and over time.

Determining creep equations of anisotropic media with different creep resistances in tension and compression were constructed in [1, 2]. Aspects of the elastic deformation of variable-modulus anisotropic materials were examined in [3, 4].

The approach proposed and substantiated in [5] for the description of creep of materials with a variable resistance to creep is extended here to the elasticity of anisotropic variable-modulus media.

1. For the elastic potential

$$W = \sigma_e^2/2$$

the equivalent stress σ_e ($\sigma_e \geq 0$) is taken on the basis of linear $\sigma = b_{ij}\sigma_{ij}$ and quadratic $\sigma_0^2 = a_{ijk\ell}\sigma_{ij}\sigma_{k\ell}$ compatible invariants of the stress tensor σ_{ij} and tensors of elastic anisotropy b_{ij}, $a_{ijk\ell}$, i.e., in the form $\sigma_e = \sigma + \sigma_0$. Then the components of the strain tensors $\varepsilon_{ij} = \partial W/\partial\sigma_{ij}$ and the stresses prove to be connected to each other by the following quasi-linear relations in anisotropic variable-modulus media:

$$\varepsilon_{ij} = \sigma_e(a_{ijkl}\sigma_{kl}/\sigma_0 + b_{ij}). \tag{1.1}$$

The symmetry of $a_{ijk\ell}$ and b_{ij} follow from the symmetry of the tensors $\sigma_{ij} = \sigma_{ji}$ and $\varepsilon_{ij} = \varepsilon_{ji}$, i.e.,

$$b_{ij} = b_{ji},\ a_{ijkl} = a_{jikl} = a_{ijlk} = a_{klij}.$$

Thus, in determining equations (1.1) the number of unknown constants $a_{ijk\ell}$ is reduced to 21 and the number of constants b_{ij} is reduced to six. Due to the invariance of the expressions for σ and σ_0, equations (1.1) retain their form in the transition to a new coordinate system. Here, the tensors b_{ij} and $a_{ijk\ell}$ are transformed in accordance with the customary rules for second- and third-rank tensors:

$$b'_{ij} = \alpha_{mi}\alpha_{nj}b_{mn},\quad a'_{ijkl} = \alpha_{mi}\alpha_{nj}\alpha_{pk}\alpha_{ql}a_{mnpq}. \tag{1.2}$$

Here, α_{ij} are the direction cosines of the angles between the principal axes x_i and the new axes x'_j.

For classical media, the linear invariant $\sigma = 0$ and equations (1.1) are transformed into the well-known relations $\varepsilon_{ij} = a_{ijk\ell}\sigma_{k\ell}$.

If an orthotropic variable-modulus medium is examined in a coordinate system with axes which coincide with the principal directions of anisotropy, the physical relations (1.1) can be written in the form

$$\varepsilon_{11} = \sigma_e\left(\frac{a_{1111}\sigma_{11} + a_{1122}\sigma_{22} + a_{1133}\sigma_{33}}{\sigma_0} + b_{11}\right),$$

$$\varepsilon_{12} = 2\sigma_e\frac{a_{1212}\sigma_{12}}{\sigma_0}(1, 2, 3),\quad \sigma_0^2 = a_{1111}\sigma_{11}^2 + 2a_{1122}\sigma_{11}\sigma_{22} + 2a_{1133}\sigma_{11}\sigma_{33} + \tag{1.3}$$

$$+ 2a_{2233}\sigma_{22}\sigma_{33} + a_{2222}\sigma_{22}^2 + a_{3333}\sigma_{33}^2 + 4a_{1212}\sigma_{12}^2 + 4a_{1313}\sigma_{13}^2 + 4a_{2323}\sigma_{23}^2,$$

$$\sigma = b_{11}\sigma_{11} + b_{22}\sigma_{22} + b_{33}\sigma_{33}.$$

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The following relations are valid for a variable-modulus isotropic medium:

$$b_{11} = b_{22} = b_{33} = B^*,\ a_{1111} = a_{2222} = a_{3333} = A^*,$$
$$a_{1212} = a_{1313} = a_{2323},\ a_{1122} = a_{1133} = a_{2233} = C^*,$$
$$a_{1111} = 2a_{1212} + a_{1122},$$

making it possible to obtain the following equations from (1.3):

$$\varepsilon_{ij} = \sigma_e \left[\frac{(A^* - 2C^*) J_1 \delta_{ij} + 2C^* \sigma_{ij}}{\sigma_0} + B^* \delta_{ij} \right],$$

where δ_{ij} is the Kronecker delta; $J_1 = \sigma_{ij}\delta_{ij}$; $J_2 = (\sigma_{ij}\sigma_{ij} - J_1^2)/2$; $\sigma = B^*J_1$; $\sigma_0^2 = A^*J_1^2 + 4C^*J_2$. Thus, the physical relations are written in this case on the basis of the first J_1 and second J_2 invariants of the stress tensor and three parameters A^*, B^*, and C^* of the material.

Thus, proposed determining equations (1.1) have sufficient generality and include several relations for different media.

Let us compare theoretical results following from (1.3) with some experimental data. To do this, we will examine SVAM glass-fiber-reinforced plastic characterized by a ratio of 5:1 (an orthotropic variable-modulus material). The elastic moduli in the principal anisotropy directions 1, 2 of this material (in kgf/mm^2) were found experimentally in [6]: $E_1^t = 5220$, $E_2^t = 1990$, $E_1^c = 5450$, $E_2^c = 3250$. Here the superscript t corresponds to tension and the superscript c corresponds to compression. Then in coordinate axes coincident with the directions 1, 2 we can determine the constants from the formula

$$a_{1111} = [(E_1^t)^{-1/2} + (E_1^c)^{-1/2}]^2/4, \quad b_{11} = [(E_1^t)^{-1/2} - (E_1^c)^{-1/2}]/2\ (1.2),$$

i.e., $a_{1111} = 1.875 \cdot 10^{-4}$ mm^2/kgf, $a_{2222} = 3.992 \cdot 10^{-4}$, $b_{11} = 1.476 \cdot 10^{-4}$ $(kgf/mm^2)^{-\frac{1}{2}}$, $b_{22} = 2.438 \cdot 10^{-3}$. Study [6] also reported experimental elastic moduli in tension and compression for flat specimens oriented at an angle $\pi/4$ to the axes 1, 2 (kgf/mm^2); $E_{45}^t = 1510$, $E^c_{45} = 1720$. Taking E^t_{45} as well as E^t_1, E^t_2, E^c_1, E^c_2 for the base data, we will attempt to predict the elastic modulus E^c_{45} and we will compare it with the experimental value. In fact, for coordinate axes rotated by $\pi/4$ relative to the directions 1, 2 we write

$$\sqrt{a'_{1111}} + b'_{11} = (E_{45}^t)^{-1/2}, \quad \sqrt{a'_{1111}} - b'_{11} = (E_{45}^c)^{-1/2}.$$

Here the prime denotes values of the parameters in the new coordinate system. Since $b'_{11} = (b_{11} + b_{22})/2$ in accordance with (1.2), we can find $a'_{1111} = [(E_{45}^t)^{-\frac{1}{2}} - b'_{11}]^2$ and then find $E_{45}^c = (\sqrt{a'_{1111}} - b'_{11})^{-2}$. Performing these calculations, we obtain a theoretical value $E_{45}^c = 1866$ kgf/mm^2. The agreement with the experimental value $E_{45}^c = 1720$ kgf/mm^2 can be considered satisfactory in the given example.

The comparison made between the theoretical and experimental results cannot be considered adequate for substantiation of the proposed determining equations. However, by virtue of the absence of experimental data in a complex stress state for variable-modulus materials [7], we will limit ourselves to this comparison. It does still to some degree validate the original theoretical premises. It should be noted that theoretical results which agreed satisfactorily with different empirical findings for a two-dimensional stress state were obtained earlier [5] in analyzing the creep of anisotropic and isotropic materials with different resistances in tension and compression by using an approach similar to that employed here. This fact also supports the validity of the use of physical relations (1.1), which also have the requisite generality and simplicity.

2. We will examine a simply connected variable-modulus anisotropic body. The displacements have been specified on part of the surface of the body, while the surface loads have been specified on the rest of the surface. The body was subjected to body forces. It is known [8, 9] that a sufficient (but not necessary) condition of the uniqueness of the solution of the elastic boundary-value problem for this body is satisfaction of the Drucker postulate in the form $\delta\sigma_{ij}\delta\varepsilon_{ij} \geqq 0$. Since nonsatisfaction of this postulate sometimes means that the solution is not unique [10], let us determine the limitations which this condition places on the parameters in equations (1.1).

We then obtain

$$\delta\varepsilon_{ij} = \delta\sigma_e (a_{ijkl}\sigma_{kl}/\sigma_0 + b_{ij}) + \sigma_e (a_{ijkl}\delta\sigma_{kl}\sigma_0 - a_{ijkl}\sigma_{kl}\delta\sigma_0)/\sigma_0^2.$$

We subsequently form the convolution

$$\delta\sigma_{ij}\delta\varepsilon_{ij} = \delta\sigma_e (a_{ijkl}\sigma_{kl}\delta\sigma_{ij}/\sigma_0 + b_{ij}\delta\sigma_{ij}) + \sigma_e (a_{ijkl}\delta\sigma_{ij}\delta\sigma_{kl}\sigma_0 - a_{ijkl}\sigma_{kl}\delta\sigma_{ij}\delta\sigma_0)/\sigma_0^2. \quad (2.1)$$

Using the relations

$$\delta\sigma_e = \delta\sigma + \delta\sigma_0, \ \delta\sigma = b_{ij}\delta\sigma_{ij}, \ \sigma_0\delta\sigma_0 = a_{ijkl}\sigma_{kl}\delta\sigma_{ij},$$

we rewrite Eq. (2.1) in the form

$$\delta\sigma_{ij}\delta\varepsilon_{ij} = (\delta\sigma_e)^2 + \sigma_e [a_{ijkl}\delta\sigma_{ij}\delta\sigma_{kl} - (\delta\sigma_0)^2]/\sigma_0. \quad (2.2)$$

Then considering the requirement of positive determinacy of the quadratic form $\sigma_0^2 = a_{ijkl}\cdot \sigma_{ij}\sigma_{kl}$, we have the inequality

$$a_{ijkl}S_{ij}S_{kl} > 0, \ S_{ij} = \sigma_{ij}\delta\sigma_0 - \delta\sigma_{ij}\sigma_0,$$

from which follows the condition

$$a_{ijkl}\delta\sigma_{ij}\delta\sigma_{kl} > (\delta\sigma_0)^2.$$

Using this relation and $(\delta\sigma_e)^2 \geqq 0$, $\sigma_0 > 0$, and the requirement, natural in all physical equations, of nonnegativity of the equivalent stress $\sigma_e \geqq 0$, we conclude from (2.2) that $\delta\sigma_{ij}\delta\varepsilon_{ij} \geqq 0$. It should be noted that the inequality $\sigma_e = \sigma + \sigma_0 \geqq 0$ is not always satisfied in numerous practical calculations for different materials in different cases. Thus, despite the fact that this inequality should be regarded as a certain limitation on the parameters in the proposed relations, it can be argued that the Drucker postulate does not impose severe requirements on the use of the given determining equations.

3. Insufficient attention is given in the literature to study of the stress—strain state in structures made of anisotropic variable-modulus materials. This has to do with the complexity of the physical relations that are used, as well as with the inadequate development of numerical methods of solving the nonlinear boundary-value problems which arise.

The theory of shells made of anisotropic variable-modulus materials is currently of great practical value. Such materials include plastics reinforced with glass or carbon fibers or metal. However, only certain problems for shells of cylindrical, conical, and spherical form have been solved so far [11, 12]. Shells of more complex shape have not been examined. The methods of solution used here are generally difficult to extend to shells of another type.

Presented below with sufficient generality is a formulation and method of solution of boundary-value problems for toroidal shells made of anisotropic variable-modulus materials.

We will examine a toroidal shell in a system of curvilinear orthogonal coordinates α, β, and z, which coincide with the directions of principal curvature. Here α is the angle between the axis of rotation and a normal to the generatrix of the torus; β is the circumferential coordinate; z is the normal coordinate, reckoned in the direction of the external normal to the generatrix of the shell. The loading is assumed to be axisymmetrical. The material of the torus is assumed to be orthotropic and of a variable modulus. The principal directions of anisotropy 1, 2, and 3 coincide with the directions α, β, and z. The physical state of the material is determined by the values of the tensors a^*_{ijkl}, b^*_{ij} entering into law (1.3).

Since allowing for the variable modulus of materials of shells leads to a nonlinear problem, the problem must be formulated in accordance with the linearization scheme that will be used subsequently. In connection with this, instead of a torus made of a specific material with the parameters a^*_{ijkl}, b^*_{ij}, we will examine the same type of toroidal shell made of materials characterized by the tensors $a_{ijkl} = a^*_{ijkl}$, $b_{ij} \in [0, b^*_{ij}]$. We then formulate the boundary-value problems for a class of shells made of certain anisotropic materials having the same values of a_{ijkl} and a different tensor b_{ij}. The geometric dimensions and loads are the same for all of the tori. We introduce the parameter t, which determines the different physical state of the materials of the shells being examined.

We write the static equations in the form

$$K\frac{d\dot{T}_{11}}{d\alpha}+C(\dot{T}_{11}-\dot{T}_{22})+K\dot{Q}_{11}+\dot{q}_1=0,$$
$$K\frac{d\dot{Q}_{11}}{d\alpha}+C\dot{Q}_{11}-K\dot{T}_{11}-\frac{\dot{T}_{22}\sin\alpha}{B}+\dot{q}_3=0,\tag{3.1}$$
$$K\frac{d\dot{M}_{11}}{d\alpha}+C(\dot{M}_{11}-\dot{M}_{22})-\dot{Q}_{11}=0,$$

where $K = 1/R$, $B = d + R\sin\alpha$, $C = \cos\alpha/B$; R and d are dimensions of the shell (Fig. 1); T_{11} and T_{22} are axial forces; M_{11} and M_{22} are bending moments; Q_{11} is the shearing force; q_1 and q_3 are surface loads in the directions α and z; the dots denote differentiation with respect to the parameter t.

The kinematic relations have the form

$$\dot{\varepsilon}_{11}=\dot{e}_1+z\dot{\varkappa}_1(1,2);\tag{3.2}$$

$$\dot{e}_1=Kd\dot{u}/d\alpha+K\dot{v},\ \dot{e}_2=C\dot{u}+\dot{v}\sin\alpha/B,$$
$$\dot{\varkappa}_1=Kd\dot{\theta}/d\alpha,\ \dot{\varkappa}_2=C\dot{\theta},\ \dot{\theta}=-Kd\dot{v}/d\alpha+K\dot{u},\tag{3.3}$$

where e_1, e_2 and κ_1, κ_2 are the strains and the changes in the curvature of the middle surface; u and v are the displacements in the directions α and z; θ is the angle of rotation of the normal to the meridian.

We will obtain physical relations for the investigated class of toroidal shells on the basis of equations (1.3) written for a two-dimensional stress state. In connection with this, we concretize the parameter t. Here we consider that the components $a_{ijk\ell}$ in (1.3) do not depend on t and that the following equations are valid for the components b_{ij}:

$$b_{11}=t\,\mathrm{sign}(b_{11}^*),\quad b_{22}=t\,|b_{22}^*/b_{11}^*|\,\mathrm{sign}(b_{22}^*).$$

Thus, the argument t is determined by the expression $t = |b_{11}|$. Then having differentiated Eq. (1.3) with respect to t, we have

$$\dot{\varepsilon}_{11}=\Delta_{11}\dot{\sigma}_{11}+\Delta_{12}\dot{\sigma}_{22}+\Delta_1(1,2),\tag{3.4}$$

where

$$\Delta_{11}=(1+\chi)a_{1111}+2b_{11}(a_{1111}\sigma_{11}+a_{1122}\sigma_{22})/\sigma_0-$$
$$-\sigma(a_{1111}\sigma_{11}+a_{1122}\sigma_{22})^2/\sigma_0^3+\dot{b}_{11}^2,$$
$$\Delta_{12}=(1+\chi)a_{1122}+b_{22}(a_{1111}\sigma_{11}+a_{1122}\sigma_{22})/\sigma_0+b_{11}(a_{1122}\sigma_{11}+$$
$$+a_{2222}\sigma_{22})/\sigma_0-\sigma(a_{1111}\sigma_{11}+a_{1122}\sigma_{22})(a_{1122}\sigma_{11}+a_{2222}\sigma_{22})/\sigma_0^3+b_{11}b_{22},$$
$$\Delta_1=(a_{1111}\sigma_{11}+a_{1122}\sigma_{22})(\dot{b}_{11}\sigma_{11}+\dot{b}_{22}\sigma_{22})/\sigma_0+$$
$$+2b_{11}\dot{b}_{11}\sigma_{11}+b_{11}\dot{b}_{22}\sigma_{22}+b_{22}\dot{b}_{11}\sigma_{22}+\dot{b}_{11}\sigma_0(1,2),$$
$$\chi=\sigma/\sigma_0,\ \dot{b}_{11}=\mathrm{sign}(b_{11}^*),\ \dot{b}_{22}=|b_{22}^*/b_{11}^*|\,\mathrm{sign}(b_{22}^*).$$

The next equation follows from (3.4):

$$\dot{\sigma}_{11}=E_{11}(\dot{\varepsilon}_{11}-\Delta_1)+E_{12}(\dot{\varepsilon}_{22}-\Delta_2)\quad(1,2),\tag{3.5}$$

where

$$E_{11}=\Delta_{22}/\Delta,\ E_{12}=-\Delta_{12}/\Delta\quad(1,2),\quad \Delta=\Delta_{11}\Delta_{22}-\Delta_{12}^2.$$

Then proceeding to integral characteristics over the thickness h of the shell

$$\dot{T}_{11}=\int_{-h/2}^{h/2}\dot{\sigma}_{11}dz,\ \dot{M}_{11}=\int_{-h/2}^{h/2}\dot{\sigma}_{11}zdz\tag{1.2}$$

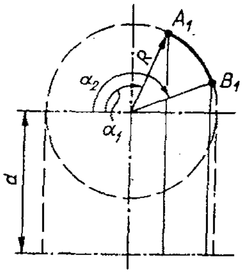

Fig. 1

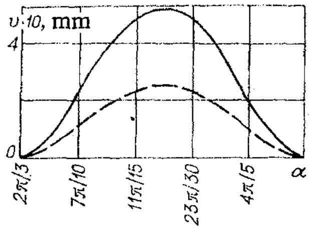

Fig. 2

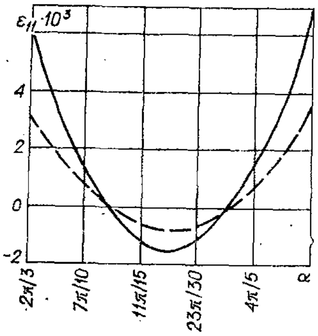

Fig. 3

and allowing for (3.2), we obtain the physical equations in the form

$$\dot T_{11} = B_{11}\dot e_1 + B_{12}\dot e_2 + A_{11}\dot\varkappa_1 + A_{12}\dot\varkappa_2 + T_1^p,$$
$$\dot M_{11} = A_{11}\dot e_1 + A_{12}\dot e_2 + D_{11}\dot\varkappa_1 + D_{12}\dot\varkappa_2 + M_1^p \quad (1,2). \tag{3.6}$$

Here

$$(B_{ij},\ A_{ij},\ D_{ij}) = \int_{-h/2}^{h/2} (1,\ z,\ z^2)\, E_{ij} bz \quad (i,j = 1,2),$$
$$(T_1^p,\ M_1^p) = -\int_{-h/2}^{h/2} (1,z)\, K_1 dz,$$
$$K_1 = E_{11}\Delta_1 + E_{12}\Delta_2 \quad (1,2).$$

Thus, the main relations for toroidal shells are represented by Eqs. (3.1), (3.3), and (3.6). By introducing the vector $\dot y = (\dot T_{11},\ \dot Q_{11},\ \dot M_{11},\ \dot u,\ \dot v,\ \dot\theta)$, we can reduce these relations to a resolvent system of nonlinear first-order differential equations:

$$K d\dot y_i/d\alpha = f_i + F_i + b_i \quad (i = \overline{1,6}), \tag{3.7}$$

where

$$f_1 = C\left(T_2^e - \dot y_1\right) - K\dot y_2,\ f_2 = -C\dot y_2 + K\dot y_1 + \sin\alpha T_2^e/B,$$
$$f_3 = C\left(M_2^e - \dot y_3\right) + \dot y_2,\ f_4 = e_1^e - K\dot y_5,\ f_5 = K\dot y_4 - \dot y_6,$$
$$f_6 = \varkappa_1^e,\ F_1 = CT_2^*,\ F_2 = T_2^* \sin\alpha/B,\ F_3 = CM_2^*,$$
$$F_4 = e_1^*,\ F_5 = 0,\ F_6 = \varkappa_1^*,\ b_1 = -\dot q_1,$$
$$b_2 = -\dot q_3,\ b_j = 0\,(j = \overline{3,6}),\ \dot\varkappa_2 = C\dot y_6,$$
$$\dot e_2 = C\dot y_4 + \dot y_5 \sin\alpha/B,\ \varkappa_1^e = [(\dot y_3 - A_{12}\dot e_2 - D_{12}\dot\varkappa_2) B_{11} -$$
$$- (\dot y_1 - B_{12}\dot e_2 - A_{12}\dot\varkappa_2) A_{11}]/\Delta^*,$$
$$e_1^e = (\dot y_1 - B_{12}\dot e_2 - A_{11}\varkappa_{11}^e - A_{12}\dot\varkappa_2)/B_{11},$$
$$\varkappa_1^* = (T_1^p A_{11} - M_1^p B_{11})/\Delta^*,\ T_2^* = B_{12}e_1^* + A_{12}\varkappa_1^* + T_2^p,$$
$$M_2^* = A_{12}e_1^* + D_{12}\varkappa_1^* + M_2^p,\ e_1^* = -(A_{11}\varkappa_1^* + T_1^p)/B_{11},$$
$$\Delta^* = D_{11}B_{11} - A_{11}^2,\ T_2^e = B_{22}\dot e_2 + B_{12}e_1^e + A_{22}\dot\varkappa_2 + A_{12}\varkappa_1^e,\ M_2^e =$$
$$= A_{22}\dot e_2 + A_{12}e_1^e + D_{22}\dot\varkappa_2 + D_{12}\varkappa_1^e.$$

Equations (3.7) are augmented by homogeneous boundary conditions with $\alpha = \alpha_1$ and $\alpha = \alpha_2$:

$$\dot y_k(\alpha_1)\gamma_k + \dot y_{k+3}(\alpha_1)(1 - \gamma_k) = 0 \quad (k = \overline{1,3}),$$
$$\dot y_k(\alpha_2)\gamma_{k+3} + \dot y_{k+3}(\alpha_2)(1 - \gamma_{k+3}) = 0 \quad (k = \overline{1,3}). \tag{3.8}$$

The file of γ_i ($i = \overline{1,6}$) determines the type of boundary conditions.

The above-formulated nonlinear boundary-value problem (3.7), (3.8) should be examined together with the Cauchy problem for differential equations (3.5) and the relations

$$dy_i/dt = \dot{y}_i \qquad (i = \overline{1,6}). \tag{3.9}$$

Here the argument t changes from zero to the value $t_* = |b^*_{11}|$. The initial conditions for (3.5), (3.9) are determined by the solution of the linear boundary-value problem for a torus made of an anisotropic constant-modulus material. This problem is written on the basis of (3.7), (3.8) if we examine the quantities themselves rather than their rates and if we assume that $T_1^c = T_2^c = M_1^c = M_2^c = 0$ and assign the coefficients A_{ij}, B_{ij}, D_{ij} their classical values [13]. Solving this boundary-value problem numerically by the stable method of discrete orthogonalization of Godunov, we find the components y_j ($j = \overline{1, 6}$) and the stresses for t = 0. We then integrate the initial problem for (3.5), (3.9) by the Kutta–Merson method, i.e., if we write system (3.5), (3.9) in the form

$$dR_k/dt = \Phi_k(R_j, t), \; \mathbf{R} = \{\sigma_{11}(\alpha, z), \sigma_{22}(\alpha, z), y_1(\alpha), y_2(\alpha), \ldots, y_6(\alpha)\},$$

then the sought functions at each point of the shell for the next moment t + Δt are calculated from the formulas

$$R_k(t + \Delta t) = R_k(t) + (r_1^k + 4r_4^k + r_5^k)/2 + o(\Delta t)^5,$$

where

$$\begin{aligned}
r_1^k &= \Phi_k[t, R_j(t)]\,\Delta t/3,\\
r_2^k &= \Phi_k\left[t + \Delta t/3,\; R_j(t) + r_1^j\right]\Delta t/3,\\
r_3^k &= \Phi_k\left[t + \Delta t/3,\; R_j(t) + (r_1^j + r_2^j)/2\right]\Delta t/3,\\
r_4^k &= \Phi_k\left[t + \Delta t/2,\; R_j(t) + 3(r_1^j + 3r_3^j)/8\right]\Delta t/3,\\
r_5^k &= \Phi_k\left[t + \Delta t,\; R_j(t) + 3(r_1^j - 3r_3^j + 4r_4^j\right]\Delta t/3.
\end{aligned}$$

Here, calculation of the right sides requires fivefold solution of boundary-value problems (3.7), (3.8) by Godunov's discrete orthogonalization method. Such a combination of the above two numerical methods is used throughout the range of integration over the argument t and is the basis of the proposed approach to the design of shells made of variable-modulus materials.

The initial step Δt is assigned, and its subsequent values are chosen automatically on the basis of the condition that the cumulative error in the calculation

$$\varepsilon = \max_k (r_1^k - 9r_3^k/2 + 4r_4^k - r_5^k/2)$$

does not exceed a certain specified value δ. Here we use the following criterion for the change in the step: If ε > δ, then the step Δt decreases by a factor of two and the calculation is repeated; if ε < δ/32, then the step is doubled and the calculation is continued.

The integrals over the thickness of the torus are determined from the Simpson formula. The numerical solution is completed after the value t_* is reached. The stresses and the components y_k corresponding to this value are the solutions found, since they are sought for a toroidal shell with the parameters $a^*_{ijk\ell}$, k^*_{ij}.

As an example, we will examine a toroidal segment (see Fig. 1) with the dimensions h = 10 mm, R = 232 mm, d = 400 mm, $\alpha_1 = 2\pi/3$, and $\alpha_2 = 5\pi/6$. The shell is under a uniform internal pressure $q_3 = 0.4$ kgf/mm². Both edges of the torus, corresponding to points A_1 and B_1 in Fig. 1, are fixed. The material of the shell is glass-fiber-reinforced plastic with elastic moduli (kgf/mm²) $E_1^t = 6000$, $E_1^c = 2000$, $E_2^t = 3000$, $E_2^c = 1500$ and a Poisson's ratio $\nu_1^t = 0.25$, corresponding to the parameters $a_{1111} = 3.11 \cdot 10^{-4}$ mm²/kgf, $a_{2222} = 4.86 \cdot 10^{-4}$, $a_{1122} = 9.76 \cdot 10^{-6}$, $b_{11} = -4.73 \cdot 10^{-3}$ (kgf/mm²)$^{-2}$, $b_{22} = -3.78 \cdot 10^{-3}$.

Figure 2 shows the distribution of the normal displcement with regard to the generatrix of the torus. Figure 3 illustrates the change in the strain ε_{11} on the inside surface of the toroidal segment. These relations are shown by the solid lines, while the dashed lines show the analogous results for a torus made of an anisotropic constant-modulus material with the constants $E_1^t = E_1^c = 6000$, $E_2^t = E_2^c = 3000$, $\nu_1^t = 0.25$. It can be seen that the difference in moduli has a significant effect on the results.

In the shell calculations we took 55 points of the generatrix (eleven of which were orthogonalization points) and seven points through the thickness. Integration over the parameter t was performed with an accuracy which made it possible to retain four reliable significant digits for the stresses. The above data were established during the numerical experiments.

The example took 5 min to solve on a BÉSM-6 computer by the above-described algorithm.

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